A ZENO'S PARADOX:
Achilles and the Tortoise

Perhaps Zeno's paradoxes are the most famous in history. In general, philosophers and historians of Mathematics have understood them as paradoxes against movement, in the sense that Zeno and the school of Elea would have used them to deny the reality of movement, demonstrating its impossibility from a logical point of view (motion in itself would imply logical contradictions). If this were so, one could argue that the world 'real' could not be understood by rational thought, or that truth should be purely ideal and phenomena just misleading appearances, etc., but this is not the problem of Zeno's paradoxes against movement. Some people took seriously Zeno's arguments (and do so today, as well, so that from time to time someone would come up with a solution of the paradox of Achilles and the Tortoise, as if the problem was justify the movement - not the search for some error in reasoning ...).

Before proceeding, let's see *Achilles and the Tortoise*.

Achilles (a champion in the races) is chasing the Tortoise, to which he has given a head start. He reaches the point where the Tortoise started the race, but in the meantime it has travelled a certain distance. Achilles covers this distance and again, in the meantime, the Tortoise runs a further distance. So, when Achilles arrives at a position already reached by the Tortoise, it reaches an even more advanced, and Achilles will never overtake the Tortoise. This reasoning is tantamount to construct an infinite number of points, such that if ABC are three consecutive points in the series, the Turtle covers the distance BC, while Achilles runs AB. Call ' Zeno's succession ' this endless series of points, and 'Zeno's points' and 'Zeno's instants' the points and the corresponding moments belonging to the succession. To avoid difficulties, we assume that Achilles and the Tortoise run at constant speed.

To discuss the topic is better to establish some assumptions. Indeed, a paradox is a contradiction, or a statement at odds with common sense or evidence. This means that not necessarily a paradox contradicts a proposition we have formally established or shown to be true by reasoning, but it can also be the result of a logically correct and complete argument in conflict with the evidence or common sense. Therefore, we must clarify the premises from which to begin the discussion. It's implicit in Zeno's arguments that we are dealing with infinity, and I interpret this as an admission that space and time are continuous in the modern sense of the term, or at least that both are *dense* (an ordered set is 'dense' if between any two distinct elements belonging to the set there's a third). In addition, space and time contain minimum indivisible units equivalent to each other, such as points and instants. In short, it appears that the formulation of Zeno's paradox requires [at least] those two general premises, and that the distance between Achilles and the Tortoise declined uniformly over time. It should be clear that *under these assumptions there is no paradox because Zeno's reasoning does not prove the conclusion stated*. To prove that Achilles will not exceed the Turtle one should prove that *no [finite] segment is, such that it contain all the positions he reached while running after the Turtle*, or that there is no instant after any instant when Achilles reaches a position which already the Tortoise exceeded. Mathematicians would say that Zeno's sequence should diverge at infinity to prove that Achilles will never reach the Tortoise. *It is not enough Zeno's points and instants (defined as above) are infinite*. This is a point often overlooked.

In considering this paradox, one usually focuses on the infinite series of distances that Achilles should cover chasing the Tortoise. The series of positions, distances, moments and time intervals that we consider are trivial: it is obvious that each of these series consists of infinite terms, for any distance or time interval contains infinitely many points, or, respectively, instants. This is true also for the entire segment that Achilles has to pass before overtaking the Tortoise. This paradox is not more paradoxical that to cover any distance, passing through an infinity of [dimensionless] points.

Even more obvious is that the point at which Achilles reaches the Tortoise can not belong to the Zeno's succession: this includes only the points that the Turtle has already passed running before Achilles and it is clear that never Achilles will reach the Tortoise if you consider just these points: the absurd Zeno's conclusion is already implicit in the choice of an appropriate set of points. Thus, Zeno's paradox has a dialectical nature: it belongs to Rhetoric. In the historical context of ancient Greece there was not a clear distinction between [formal] Logic and Dialectic, and the discussion between two speakers proceeded normally such as in Plato's dialogues, where a point of view (that of Socrates, of course) wins on the other, by showing that the later leads to a contradiction. The definition of a clear, systematic distinction between correct reasoning and sophism is evident in Aristotle; before him, deductive reasoning and dialectic live mixed together. From this point of view, the paradoxes against movement seem to have a psychological character. Paradox is a fundamental tool of dialectical method.

However, there is also a mathematical aspect that in itself implies something of a paradox, because Zeno's argument involves infinite sums of terms tending to zero. In order to clarify this question, let's consider the problem from a mathematical point of view. We began by examining the sequence of points such that its first point is the location from which the Tortoise starts running, the second one the position the Tortoise reaches when Achilles reaches the first point, *and so on*, x_n is the position of the Turtle at a time when Achilles reaches x_{n-1}, then for all terms we have $x_n > x_{n-1}$. There are infinite terms x_n, and *this is the only result of Zeno's argument*, and it seems awfully trivial. As seen earlier, to prove his paradoxical conclusion must be $\lim_{n\to+\infty} x_n = \infty$, that we can't get only knowing that Achilles must pass infinite points to reach the Tortoise.

Each point x_n corresponds to a certain instant t_n, so we also have a succession (an infinite set in one-to-one correspondence with the natural series) of general term t_n. In addition, we also have an infinite sequence of consecutive distances δx_n with $\delta x_n = x_{n+1} - x_n$ and an infinite sequence of consecutive time intervals $\delta t_n = t_{n+1} - t_n$. If Achilles can not pass the Tortoise, as Zeno says, then the sum of all terms of both sequences should be infinite (the distance that Achilles covers before reaching the Turtle would be infinite), else both sums must be finite. To prove Zeno's conclusion, which is actually a paradox, it is necessary that the first case is the real one. Here there is indeed a mathematical difficulty, because it may seem strange that a sum of infinite terms can be finite, even when the sequence of these terms is decreasing. In fact, this problem is easily solved (by the way: there is a very common tendency to use the limit of a series of numbers to solve Zeno's paradoxes. There is no real necessity to proceed in this way, but infinity implies actual paradoxes in many cases, so it may be useful to employ advanced tools in order to eliminate any doubt). In this case, the infinite leads to the following question: how can Achilles reach the Tortoise, if he has to go through infinitely many points? Our mind is focused on 'infinity' and loses itself imagining that Achilles must traverse an infinity of positions to reach the Turtle ... as if to perform a procedure in which each step takes time. Achilles does not build the space or time – the points x_n and the instants t_n *already exist* and no one should 'build' them - the individual terms of Zeno's series are not steps of a procedure. *The only thing we must check is whether a sum of infinite terms can be finite*.

Think of a series of consecutive segments on the same line, so that each segment is half of its immediate predecessor. The sum of all segments can not be more than twice the first. Mathematicians easily show that in the case of an infinite series the sum of all segments is just twice the first, using the concept of limit, as $\lim_{n\to+\infty} \sum_{0}^{n} \frac{1}{2^n}$ = 2. The notion of limit is also implicit in the idea of a quantity, smaller and smaller, which is tending to zero, as Mathematicians say, because using the limits all sequences and series we have considered above converge to finite limits, so to conclude the matter. But it is clear that if the sum of a finite number of distances δx_n travelled by Achilles is in any case less than twice the first, the sum of all the distances (which are an infinite number) can not be infinite.

However, even recently someone tried to resolve the paradox by proposing original solutions. One of these is to discuss the space structure on extremely small scale. The difficulty may arise when we accept that space is infinitely divisible, because that would imply the infinite, but in reality the space is not a *continuum*, and we would not need to resolve the paradox. Now, it is reasonable that at very small size the properties of space are different from the axioms that Mathematicians and Physicists commonly use, but this is not the core of the matter, since infinite divisibility of space does not imply the conclusion of Zeno's argument while, on the contrary, Zeno's argument requires the indefinite divisibility of space.

However, it may be that the significance of this pseudo-paradox is somewhat different from the traditional interpretation. If the infinite is the problem since infinity is implied by the indefinite division of space, then it may be that Zeno's arguments against motion were also meant to oppose the notion of undefined divisibility. In fact, the ancient Greek Mathematicians did not introduce the infinite so far as possible. It is very likely that Zeno's paradoxes had a close relationship with the discovery of the *incommensurables* due to the Pythagoreans, and - from this point of view - have played a by no means negligible part in Mathematics. Why the Greeks avoided the use of the infinite? The most likely answer is in the constructive character of their methods. Geometry originated from practical problems; while considering that the birth of abstract thought and deductive proof was relatively old, for many centuries, the Greeks were tied to visual representation of the figures and their methods consisted mainly of geometric constructions. Now, a construction consists always of a finite number of steps, even if it is possible in principle to repeat a passage many times as we want: thus, we operationally divide a segment into two parts, and bisect both n times, but we can not do countless times. However, nothing prevents you imagine that a segment is composed of an infinite series, the first segment of which is half of the whole segment, the second half of the first ,... the n-th the half of its predecessor. So the whole segment is ideally divided infinite times, but obviously it is physically impossible to realize such a construction. There is a strong similarity between geometric constructions and algorithms: an algorithm must be a procedure executable using a finite number of instructions in a finite time. The undefined division is operationally impossible, and in any case it may seem meaningless. However, incommensurable quantities show that a minimum segment does not exist, so *to avoid contradictions* space must be made up of infinite points with no extension. This is paradoxical, since it seems you can not build a line by putting together dimensionless objects. Running a distance, we have to go through infinitely many points, but it is not like counting to infinity in finite time, or the statements of an endless procedure, because points and instants are dimensionless. The paradox lies in composing infinitely many terms in a finite result. Indefinite division of space, dimensionless points and finite sums of infinitely many terms are unavoidable in order to formulate the paradox of Achilles and the Tortoise and they are also the tools to solve it.

We must note that Achilles can not reach the Tortoise if his speed decreases compared to that of the Tortoise, according to some special law of motion. It is better to treat the case as a movement slowed, taking the Turtle at rest while Achilles runs toward her. Assume that its relative speed V is inversely proportional to time raised to α , with $0 < \alpha < 1$:

$$V = \frac{k}{t^{\alpha}}$$

The space Achilles covers in time t is $\int_0^t \frac{k}{x^\alpha} dx = \frac{k}{1-\alpha} t^{1-\alpha}$. $\lim_{t \to +\infty} \frac{k}{1-\alpha} t^{1-\alpha} = \infty$ when $0 < \alpha < 1$, so in this case Achilles will not reach the Tortoise.

Finally, let's consider the problem of space structure from a conceptual point of view. Historically, our ideas about space and Geometry are derived from observation. Up to a certain point, the mathematical properties of space reflect common experience, but some have a theoretical foundation. It's the case of the infinite divisibility of space, which can be defined by the *axiom of divisibility*, which states that, given a quantity Q and for any [natural] number n, there is a submultiple S of Q such that $n\ S$ is equal to Q. In short, the "minimum submultiple" of a certain given quantity does not exist, so any quantity is divisible to infinity (this refers to segments, surfaces, volumes, angles ... we have discrete quantities in Physics as electric charge, etc.)... We must admit the infinite divisibility of space in order to avoid contradictions with the existence of incommensurable quantities, as the ratio of diagonal and side of the square.

Of course, all the geometric concepts are ideal entities: points, lines, surfaces in Geometry describing the properties of material objects, but are ideal self. Therefore, the development of Geometry and Mathematics was conducted according to the principle of non-contradiction: the structure of the whole system is a coherent set of logical relations. What is really when it comes to 'space'? We refer to a geometric space, which is a model of a system of axioms, or to the physical space whose intrinsic properties are not known completely and that we try to describe by a geometric model? It seems that the second answer is that true, since Physicists have replaced the Newtonian-Euclidean space with the curved space-time of General Relativity to gain a better theory of gravitation, so scientists have to choose the best geometric model based to experimental data. This leads to the idea that space has an inherent structure. However, in practice, because the physical space is unknowable in itself, we can only study the properties of the models we use to approximate the real space. Moreover, in each case properties, structure, relationships, etc. are ideal beings, even if it refers to something "real." Thus, *in practice*, we refer only to geometric spaces with the appropriate properties.

We have to think the "real" space, not as a definite object, but rather as a system of properties, or better a set of axioms that we can state in order to avoid contradictions between the axioms themselves and between axioms and observations. For example, Zeno's paradox shows us that the continuum is necessary to identify the point at which Achilles reaches the Tortoise. To define this point (and the corresponding moment) you can use Dedekind's postulate, ie it can be defined as the *element of separation* of two adjacent sets of points. Dedekind's postulate is applicable to all the continuous sets and back every continuum (in the mathematical sense of the word) satisfies it. This axiom says that the line is the union of two rays, such as not having points in common, so their common origin must belong to only one of the two. Consider the line of motion and the half-line that contains only all Zeno's points defined as above and all the previous ones, excluding all those which follow all Zeno's points. So, the point at which Achilles reaches the Tortoise does not belong to this half-line: it must belong to the other ray. This point is the unique *element of separation* between the set of points where Achilles is chasing the Tortoise and that of the points where he is running before her. We can also note that this definition does not imply explicitly any limit, even if the point so defined is the limit of Zeno's sequence, as a continuum is composed of all its limit points. But, from a mathematical point of view, the core of the matter is the continuum, which was already introduced in Geometry through the incommensurables. We can say that Zeno's paradox had, as regards space and motion, the same meaning as the ratio of diagonal and side of the square.

As a result of this analysis, we conclude that all reasoning about space, movement, etc. is strongly influenced by the system of axioms that we have, often implicitly or unwittingly, used in the discussion. However, the system used can not be at all arbitrary: it must satisfy the principle of non-contradiction, and contain all the conditions necessary to clearly define the problem. As discussed above, Zeno's paradox needs infinity and can not be examined without the infinite series and infinite divisibility of space. This applies to any argument involving expressions such as 'and so on ...', which implicitly refers to an infinite series of steps without using the word 'infinity'. Zeno's argument is based on the infinite and the solution also.

NOTE

1. The ancient Greek thought is partly misunderstood or unknown, especially in the case of the Presocratics. So there is likely to miss the point of some fragments and, more generally, the overall significance of the thought older. Above all, do not know enough about the true aims and implications of the forms of knowledge away from our mentality. Then one may express doubts about the meaning of the paradoxes and Dialectic in ancient Greece. Zeno's paradoxes are apparently so absurd as to suggest that they were only demonstrations of dialectical skill to impress the audience or winning verbal games against some challenger, unless you believe to be true to the traditional interpretation, that the purpose of these arguments was only to demonstrate the impossibility of the movement ... Which is nonsense, unless we remember what was said before. But it may be - and still is very reasonable to suppose - that paradoxes were mainly intended to call into question some aspects of rational thought, if carried to its logical conclusions, to induce the listener to reflect on thought and reality. In any case, just what was the outcome of this method of reasoning. In this sense, the paradoxes have made a decisive contribution to the development of Mathematics.
It can also be useful to search for analogies in other civilizations that developed in parallel, like India under the influence of Buddhism, but this way you may come to conclusions very different. Dialectics and paradoxes have played an important role in Indian tradition, with different objectives and results, as happened in pre-classical Greece. Some schools of thought utilized paradoxes or unlikely examples to lead the listener or the reader beyond the conventional wisdom and ordinary perception of reality, according to the specific methods of Buddhism and Vedanta. Nagarjuna (2nd century AD) used several paradoxical examples to break the conceptual framework of common sense. See, for example, his *Fundamental Verses of the Middle Way*, where he tried to destroy the rational categories and common habits of thought, deep rooted, using only the tools of rational thought.

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