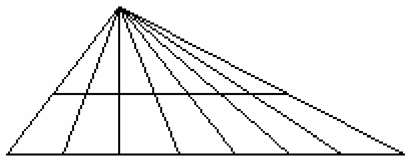
INFINITY IN MATHEMATICS
A brief introduction

Infinite sets have not been object of systematic researches by mathematicians until middle 19^{th} century. This fact is due to the difficulty in handling this subject without falling into paradoxes and contradictions. For instance, it's difficult to define how to compare two different infinities. We can consider the Galileo's paradox wondering whether natural numbers (0, 1, 2, 3…) are more numerous than their squares (0, 1, 4, 9…). At first sight, we should conclude natural numbers are "more numerous" than perfect squares (or than even numbers, or integer multiples of some natural number, or prime numbers etc…), since there are an infinity of natural numbers which are not square numbers. However, Galileo points out that each natural number corresponds to only one square number and vice-versa, according to a biunivocal (i.e. one-to-one) correspondance:

0	1	2	3	4	5	6	7	8	9
0	1	4	9	16	25	36	49	64	81

Now, two *finite* sets in one-to-one correspondence have the same number of elements. By extending this criterion to infinite sets we should conclude that natural numbers aren't more numerous than the perfect squares or the even numbers etc., contrary to the *Aristothelic principle according which the whole is greater than a part*.

There are paradoxes even in Geometry. For example, consider two unequal segments. It seems the longer one contains more points than the shorter. But trace from an external point some straight lines intersecting both of them as follows:



As we can see, each point of the shorter segment corresponds to one and only one point of the longer, therefore now it seems both segments contain the same number of points.

Such paradoxes and the difficulties involved in defining coherently the concept of infinite quantity induced mathematicians to employ the notion of infinity the less possible, until the German mathematician *G. Cantor*, in the second half of 19^{th} century, developed a first *set theory* by which to handle coherently the problem of mathematical infinity. Cantor started from the concept of set as a collection of distinct objects and defined "infinite set" every set which can be put in an one-to-one correspondence with an its *proper subset* i.e. a subset *non-containing all the objects belonging to the set itself* (such a definition seems strange, but in set theory every set is a subset of itself). For instance, $\{2, 6, 7, 10\}$ is a "proper subset" of $\{1, 2, 4, 6, 7, 10\}$. Evidently, the number of elements belonging to a proper subject of a given *finite* set I is always strictly less than the number of elements belonging to I. But for deciding if two *infinite* sets have the same number of elements (in Maths, it is said they "are idempotent" or "have same cardinality") we must define once and for all what "to have the same number of elements" means, so avoiding ambiguities and contradictions. The Cantor's definition is that two sets have the same number of members if it is possible establish an one-to-one correspondence between the elements of A and those of B, so the number of naturals and the number of even numbers are the same. But the set of the even numbers is a proper subset of the set of the natural numbers, therefore the whole set and a proper *infinite* subset can be put in biunivocal correspondence. This is a peculiar property of the infinite sets.

At this stage, a problem takes place into Maths: have infinite sets the same cardinal number, i.e. are they all in a one-to-one correspondence between each other? For example: have the set of rational numbers (i.e. the fractions) and the set of natural numbers the same cardinality? Apparently, the answer seems to be *no*, because of the differences between the two sets (e.g. between two rationals a third always does exist, but this property doesn't apply to naturals); nevertheless we can prove that an one-to-one correspondence can be established between the set Q of the rationals and the set N of the naturals, but to reach this result we need to order the set Q not according with greatness but according with a scheme like the following:

1/1	½	1/3	1/4	1/5	....	1/n	......		
2/1	2/2	2/3	2/4	2/5	....	2/n	......		
3/1	3/2	3/3	3/4	......	....	3/n	......		
4/1	4/2	4/3	4/4	......	......	4/n	......		

The order according to which to sort the set Q is $\frac{1}{1}$; $\frac{1}{2}$; $\frac{2}{1}$; $\frac{1}{3}$; $\frac{2}{2}$; $\frac{3}{1}$; $\frac{1}{4}$; $\frac{2}{3}$; $\frac{3}{2}$; $\frac{4}{1}$ etc., i.e. the infinite 2-dimensional matrix of all the rationals, built as above, is decomposed in a succession of finite diagonals each of them begins with a term $\frac{1}{n}$ (1^{st} row, n-th column) and finishes with a term $\frac{n}{1}$ (n-th row , 1^{st} column of the matrix); in this way the same number (e.g. $\frac{n}{n}$ = 1) turns up infinite times, but this doesn't modify the conclusion of the reasoning: by this way each rational number $\frac{m}{n}$ is coupled with one and only one natural number; therefore *according to Cantor's criterion natural numbers and rational numbers have the same cardinality*, even if there is an infinity of rational numbers such as $\frac{2}{3}$, $\frac{5}{4}$ etc. which are not equal to any positive integer. [1]

It remains to examine the set of all the real numbers, i.e. $\Re$. To establish whether $\Re$ and N have the same number of elements we start from the *infinite decimal representation of a real number*, by which every real number can be represented by one infinite succession of digits and vice-versa any infinite succession of digits separated by a comma defines one real number. Actually, it's enough to examine the reals included between 0 and 1; indeed, if this set and N had not the same cardinal number then even $\Re$ and N will not have same cardinality.

Now suppose $\Re$ is in biunivocal correspondence with N; then we can think there is a *first* real number, a *second* real number, a *third* one and so on; i.e. we can imagine an *infinite countable list of all real numbers*, no one excluded, like for the set of rationals.

0, $a_1\, b_1\, c_1\, d_1$.........
0, $a_2\, b_2\, c_2\, d_2$........
0, $a_3\, b_3\, c_3\, d_3$........
....
....
0, $a_n\, b_n\, c_n\, d_n$.........
....

The letters indicate the digits in succession and n indicates the real number corresponding to n .

But this hypothesis is contradicted by determining *at least one real number which cannot belong to this list*. Indeed, by succeeding in finding even only one real number not belonging to the list, we prove this numbering isn't complete. We can construct a real number different from each number of the listing by stating its first decimal digit is different from a_1, the second from b_2, the third from c_3, the *n*-th different from the *n*-th digit of the real number corresponding to the natural number n, so that its digits are systematically different from those of the diagonal a_1 b_2 c_3 (indeed this procedure is known as "*diagonal method*"). The real number so defined is different from all the numbers of the list, in contradiction with the hypothesis according to which the list contains all the real numbers.
Therefore, *the cardinality of the set of real numbers is greater than that of natural numbers*; in a less formal language, reals are "more numerous" than naturals.

Cantor himself introduced the notion of "cardinality of a countable infinity" to denote the "number" of all naturals, and of "cardinality of the continuum" to denote the "number" of all reals, and assigned to them the symbols $\aleph_0$ ("aleph-null") and **C** respectively.

As we have seen above, the *cardinality* or *"power"* of a set is analyzed via one-to-one correspondences, since it's enough to find a one-to-one correspondence between two sets to prove they are idempotent. But not all relations between two sets are biunivocal. A special ordering of rationals is needed in the diagonal method. However, *the power of a set is independent of the order of its elements* – it's an *intrinsic* property of the set – and must not be confused with other properties as *density.* For example, *Q* is a "dense" set while *N* isn't, but they have same cardinality.

The relation between $\aleph_0$ and C can be established via the notion of power *set.* Given a *finite* set *A*, the power set of *A* is the set whose elements are all the subsets of *A*, including the set *A* itself and the empty set $\varnothing$. For example, the power set of $\{1,2,3\}$ is $\{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\varnothing\}$.
If a *finite* set has *n* elements then its power set contains 2^n elements [2]. Extending this rule to infinite sets, the power set of N will have $2^{\aleph_0}$ ("2 raised to $\aleph_0$") elements, etc.; hence, *the power set of a given infinity I has greater cardinality than I.* Therefore, starting from the smallest infinity i.e. $\aleph_0$ we get an *infinity of infinities*, each of them is 2 raised to the cardinal number of the immediate predecessor.

$2^{\aleph_0}$ is the cardinal number of continuum, i.e. $2^{\aleph_0} = C$. To prove this statement intuitively, we consider the one-to-one correspondence between the reals and the successions of binary digits (neglect the comma [3]). Every succession defines a series $\sum_{n=1}^{+\infty} d_n \cdot 2^{-n}$, which is a real between 0 and 1 and d_n is 0 or 1. The successions of the terms d_n are $2^{\aleph_0}$ (there are 2^n different series with repetition of *n* elements equal to 0 or 1).

Mathematicians have tried to establish if some cardinal number exists between $\aleph_0$ and C. The conjecture, according to which such a cardinal number doesn't exist, is known as "*continuum hypothesis".* [3]

NOTES

1. The analytical law of this correspondence can be found as follows.
In each diagonal the sum of numerator and denominator is a constant. The number of diagonals with the constant less than or equal to a given value n is n -1, and the terms belonging to a diagonal with constant k are k – 1. The number of terms belonging to the first n diagonals is $\sum_{k=1}^{n-1}(k-1) = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$, so $\frac{1}{n}$ corresponds to $\frac{(n-1)n}{2}+1$ $= \frac{n^2-n+2}{2}$ and $\frac{m}{n}$ to $\frac{(n+m-1)^2-(n+m-1)+2}{2}+m-1 = \frac{n^2+m^2+2mn-3n-m+2}{2}$.
Therefore, the one-to-one relation between Q and N is

$$\frac{m}{n} \rightarrow \frac{(n+m)^2-3n-m+2}{2},$$

counting as distinct terms all the fractions equivalent to a given $\frac{p}{q}$ with p and q prime each other.

2. Every subset of A is built by choosing elements of A. Sort the elements of A in a certain order; and assign 1 or 0 to each element, depending on if this belongs or not to a given subset. So every subset is defined by a series of n terms equals to 1 or 0. Since a single term takes two values, there are 4 = 2^2 different dispositions for a couple of terms, 8 = 2^3 for three terms, etc…the dispositions with repetition of an ordered series containing n terms are 2^n .

3. Let $\{a_i\}$ with $i \geq 0$ be the set of all the digits of a positive real r, neglecting the comma. $\{a_i\}$ defines one and only one succession of general term $q_n = \sum_{i=0}^{n} a_i \cdot 10^{-i}$. The number r is given by multiplying $\lim_{n\to+\infty} q_n$ by 10^p , where p is the characteristic of r. The cardinality of $\{10^p\}$ is $\aleph_0$, so the [positive] reals are $2^{\aleph_0} \cdot \aleph_0 = 2^{\aleph_0}$. This confirms the comma is irrelevant.

4. The continuum hypothesis is coherent with ZFC set theory, but not derivable in it (P. Cohen, 1963).

Turin, August 2009 by Ezio Fornero