SPACE AND MOTION AS PROBLEMS OF CLASSICAL PHYSICS

1. Introduction

Space, time and motion are the first elements of any physical world description which wants to preserve a relationship of continuity with common experience. Although being aware of provisional and illusive nature of such an immediate perception of the physical Universe, it's impossible to approach physical sciences without starting from a description of the world as made of objects moving in space at passage of time. Historically, the development of modern Mechanics during 17 – 18th centuries has brought these premises to their extreme formal consequences, until already in the course of 19th century facts came to light, which have put the bases of the crisis in the 20th century, only partially resolved by the two theories of Relativity and Quantum Mechanics, since any "final" theory (if it will ever be elaborated) should compound two incompatible ways of seeing, the deterministic and reversible one of Relativity and the probabilistic of Quantum Mechanics.

I think it's impossible to thoroughly comprehend why the development of physical science has took in Western World that radical and irreversible character which has transformed the world by giving birth to the industrial civilisation and modern culture. People can give many explications of social and political nature; however, the answers to this question don't depend only on something objective, but also on cultural formation of the person who's analyzing this problem. Let's not disregard the importance of Mathematics, already developed by Greeks in the forms of Geometry and perfected in the beginnings of modern age by the developments in algebra, calculus etc. Without calculation power, Physics is impossible.

But this conclusion implies that physicists transfer mathematical methods and geometric concepts in their interpretation of phenomenal world. Classical Physics is grounded on Euclidean Geometry by transforming the ideally dimensionless geometric point into the "material point", an ideal particle moving over mathematically analyzable paths. This geometrical ground of classical Mechanics is mostly evident in Galileo's and Newton's work and – as said before – has appeared unchangeable and necessary until the beginnings of 20th century, when it was understood that it doesn't exist only one "real" Geometry, but all coherent geometries are equally "real", since the principle of non-contradiction is the only necessary presupposition of any axiomatic system and coherent physical theory.

2. Frames of reference and absolute space

The principle of inertia is the first of Newton's Dynamics laws; its statement is as follows: Every body persists in a state of rest or uniform straight motion unless an external force is applied to it ("external force" means the resultant of all the external forces acting on a body). This statement is manifestly incomplete. Compared to whom or what a movement can be defined as straight and uniform? In order to describe the motion of a body we have to measure its velocity and acceleration relatively to some other body or rigid system or to ourselves; e.g. while studying pendulum oscillations inside a laboratory we suppose the laboratory and we ourselves to be motionless and refer the pendulum motion to the vertical line through the hanging point; so doing we neglect Earth's daily rotation, Earth's yearly revolution around the Sun, etc.; *hence the description we get is relative to a given observer* while being quite different if referred to Earth's centre or to another body of the Universe. Are all the possible ways to describe motion equally arbitraries? Or does a preferred point of view corresponding to only one correct description exist, while all the others are, in a certain sense, apparent?

According to Newton, a privileged frame of reference does exist, but we aren't talking about some object or system of material bodies identifiable by direct observation; motion of material bodies and observers is always a relative motion; the privileged system of reference is space itself, hence we can say that a motion is "really" linear and uniform (or accelerated on the contrary) only by making reference to space, *which is "absolute" because is independent of any observer*. To understand this point of view we have to consider the Universe as an ensemble of bodies contained in a boundless, homogeneous space without a centre, whose geometric properties and existence itself don't depend on presence of matter and observers. If matter were removed, space would continue to exist unchanged; the motion of an observer doesn't influence his perception of space. Newton defined *absolute* such a space (i.e. independent from observers and bodies), *true* (in opposition to surrounding environment which we generally refer to, what is in motion with respect to absolute space) and *mathematical* (because its structure is defined by Euclid's geometry, the only believed true in his days); briefly, we'll say *absolute (*or *Newtonian*) *space.*

3. Principle of Classical Relativity

At this point, two problems arise: 1. how can we verify or at least justify the Newton's theory of space? 2. Even supposing Newton was right, how could we evaluate our "motion state" relative to absolute space?
According to Newton, a deep analysis of the second point leads us to verify the first; hence we proceed by examining the motion state of an object with respect to space.
First, we can note how this problem isn't at all abstract or negligible. Indeed, to define a special relationship between motion and space may seem useless: why should we not consider only motion relatively to material objects by accepting as equally valid all the possible descriptions, but taking into consideration case by case the most opportune one? However, in practise we apply just this rule: when we say "the speed of a car is 50 km/h", we refer to Earth's surface, without asking what "real" velocity is with respect to a some hypothetic "absolute space", at all needless in such a context. Really, the history of Astronomy contradicts such a thesis: if all motions are relative and the system of reference is indifferent, why should we consider planets in revolution around the Sun and not around Earth? Moreover, why do we say that the Earth is rotating on its axis in 24 hours? Evidently, the different systems of reference don't seem to be equally valid; on the contrary they form a hierarchy. The prevalence of the heliocentric hypothesis on the geocentric demonstrates historically that some descriptions are "more true" than others. Hence it's possible that there exists an "absolutely true" description consisting in the relationship with the space itself, since such a description could not involve single materials bodies. However, apart from astronomical considerations (significant by themselves but not yet sufficient), a fundamental reason exists in favour of Newton's theory or every other theory which, although disagreeing with the Newton's one in some point, preserve some a distinction between correct and incorrect descriptions, i.e. between valid and invalid observations. *This reason is the possibility in itself of Physics as an objectively meaningful science*: Physics' laws must be absolute, that is not relative to particular observers or systems of reference, but universally valid: it can be only if all the single observers agree together when describing the same phenomena; since this actually doesn't happen, *we have to make reference not to particular bodies and observers, but to an "absolute" immaterial system.* According to Newton, absolute space is the sole valid reference in which the laws of motion are meaningful. On the contrary, each observer would describe his own Universe and we could not devise a unitary theory of Universe.

This is Newton's method, but it's also possible to follow the opposite way –that is., to search for a theory under which all observers are equivalent even if their observations are disagreeing. In such a theory *laws of Physics are invariant with respect to all the possible different systems of reference*, so special frames of references are not needed. This is an optimum method to discover the *real*, universal physical laws and leads to General Relativity.

Let's follow for now Classical Dynamics.

While acknowledging that Newton's method is logically correct, it however implies in itself some serious difficulties. The first one is that Physics acquire an essentially theoretic, ideal character; in fact, the laws of Physics aren't actually universal, but apply only to observers in rest or uniform translational motion with respect to the absolute space. The second one is a natural consequence. Observations and measures are always performed inside local environments or referring to material objects: how can we know the relationship between a single body and the absolute space? In absence of a solution, Newton's theory would be useless.

Hence, we have to know how frames of reference are physically related to space.
While talking about "relation of a body or observer with absolute space" in Newtonian dynamics, we intend *three* pieces of information knowable in general through the same experimental procedures which anyone can carry out in whatever place and time, i.e. position, velocity and acceleration relative to space. Clearly, we can't define the *absolute position* of a body, firstly because of theoretical reasons: Newton's space is infinite and homogeneous and therefore without a centre (on the contrary, it was feasible in the geocentric Aristothelic system and in the heliocentric Copernican); all places in the space are indistinguishable and therefore equivalent.

Now, the problem is velocity. We observe that velocity of a given body is V relatively to a given observer; what is its velocity $\mathbf{V}$ with respect to space? Also in this case we can't define the absolute velocity of a body (although nothing prohibits to think that $\mathbf{V}$ has a defined value: but in Physics it is strongly required that a quantity is experimentally measurable; the only hypothesis that it has a defined value is inconsistent by itself if we can't measure it).
According to Classical Dynamics *we can measure only relative velocities*. To correctly analyze the problem we must introduce the concept of "closed system" as a limited, ideally mechanical isolated environment inside which no information can be retrieved from outside. The *Classical Principle of Relativity* (don't confuse it with Einstein's theory of Relativity) says: "*inside a closed system in linear and uniform motion relative to an external body it's impossible to perform an experiment to know its state of rest or motion with respect to that body*". For instance, suppose you are inside a railway wagon in straight and uniform motion with respect to Earth's surface, and ignore vibrations, jolts etc. Moreover, you cannot look outside or receive any information from external environment. This principle states that in these conditions you cannot know if you yourself and the wagon, i.e. the closed environment, are in motion or in rest with respect to Earth's surface; *a fortiori* you can't measure your absolute velocity. This ideal experiment shows how measures of velocity can only be *relative* to other bodies: in fact, every phenomenon observed inside a closed system in motion with constant velocity occurs as if it were still. Operatively, velocity relative to space is nonsense.

However, *the Classical Principle of Relativity presupposes that the concept of linear uniform motion is consistent with Newton's dynamics*, i.e. *we can experimentally establish, inside a closed system, whether the motion of the system is linear uniform or it isn't*, even if we can not measure the velocity. According to Newton's theory this distinction implies necessarily that a motion is uniform or not with respect to absolute space, since there is no need of relative measures. Newton thought to absolute space as a geometric postulate needed to justify the difference between the linear uniform motions and the not uniform ones. Observations and measurements performed by observers in linear uniform motion are all equivalent and all observers in linear uniform motion with respect to space must agree that phenomena obey the same laws.

In fact, from a logical point of view Classical Relativity Principle seems to be a corollary of Inertia Principle. Imagine an ideal wagon travelling in linear uniform motion and consider phenomena occurring inside it. Nothing inside happens to show the motion of the wagon. For example, objects put on the floor and free to move, bodies hanging from the ceiling etc. in rest relative to the wagon in a certain instant will persist in rest while the velocity of the wagon doesn't change. The motion relative to Earth's surface can't be detected *inside a closed system* because, according to Law of Inertia, all free bodies inside the system persist in straight uniform motion in absence of external forces and move with the system. Therefore, as a consequence of Inertia, an observer inside an inertial system can't recognize velocity only by performing observations inside the system in absence of external information. However, this applies also to non-inertial systems, since in no case we could know *velocity* without exploiting information from external environment; so, physical meaning of Classical Relativity seems to be more general than inertia principle. Classical Relativity is an operative principle stating what we can know only by operating inside closed systems, while 1st law is an *axiom* of motion.

Make attention to some apparent slimness. Talking about "motion", we implicitly refer to both velocity and acceleration. If the discussion concern inertial systems, "in rest" means "velocity *and acceleration* are null" , and "state of motion" refers to velocity relative to something. It's clear; but referring to accelerated systems, the principle states however that we can't know if, *in a given instant*, the velocity of a closed non-inertial system is null and, more in general, what its value is, even if we can know if the system is accelerating in absence of external information.

4. Acceleration and absolute space according to Newton

By the term *acceleration* we intend the ratio between variation of velocity per time.
The previous deductions do not apply to acceleration. Imagine again to be inside the wagon, initially in rest or in uniform translational motion relatively to Earth's surface, without retrieving any information from outside. Suppose that, starting from a certain instant, the wagon accelerate; can you detect and measure its acceleration relative to Earth? The answer is positive: especially if the acceleration is great enough, the observer himself perceives a force proportional to the acceleration pushing him in the opposite direction (if acceleration is forward, he is pushed backwards and vice-versa). Similarly, objects free to move unhindered and without friction seem to be put by themselves in motion in absence of forces whose source is internal to the wagon. Such apparent forces (defined as "inertial forces") do not appear while motion is linear and uniform. An observer inside an accelerated system will see an object, initially still with respect to the system, to fall not along the vertical line as it would happen if velocity relative to Earth's surface were constant. The resultant motion is the composition of the naturally accelerated along the vertical line, and the one due to the inertial forces, oriented in the direction opposite to the acceleration of the system.
An observer who verifies these phenomena can evaluate the acceleration relative to Earth's surface of the system which he's solidal with; can he say that the system is in acceleration relative to space? Newton's answer is *yes*, because we can prove the existence of absolute space just by observing the accelerated motions.

Really, Newton affirmed to have performed an experience which confirmed his theories, the one of the *rotating bucket*, essentially based on analyzing centrifugal forces. He filled a bucket of water and hanged it to the ceiling by a rope, then twisted the rope several times until it became rigid; he expected the surface of water to become flat again and then left the rope to unroll, placing the bucket in rotation. At the beginning, the motion is transferred only to the bucket – it takes some

time until water, thanks to the contact with internal surface of the bucket, begins to rotate. Hence, at the beginning of the experiment, water rotates with respect to the bucket. After some time, water and bucket rotate together being in rest with respect to each other. He noted that the surface of the water rose to the edges, to a greater extent the faster the rotation was.
Newton inferred that centrifugal forces pushing the particles of water far from the axis of rotation prove manifestly the *absolute* rotation (i.e. with respect to space) of water, since a *relative* rotation (i.e. with respect to the bucket) doesn't involve any observable effect like surface curvature. The centrifugal force perceived by who's turning on a spinning platform is exactly the same effect.
Rotational motion is always detected without referring to external bodies. So, we can evidence the Earth's daily rotation by observing in laboratory the rotation of the oscillation plane of a pendulum round the vertical line through the point of suspension. The daily rotation of fixed stars proves only a *relative* motion of the Earth, but since the two rotations occur in the same time, we can consider the sky of fixed stars in rest with respect to space. Hence, *in practise the fixed stars are the absolute frame of reference and the physical laws are verified only by the observers in translational uniform motion relative to the fixed stars*.

5. Mach's Principle

Several objections were opposed to Newton's theory about space and time. The discussion was focused on the claimed independence of space on matter filled into it, but until the 20th century the critic was confined within the epistemological sphere and gave no rise to alternative theories.
Analyzing the concept of absolute space, we can infer that it includes the following hypotheses: 1. it exists regardless of matter; 2. its geometry is the Euclidean one, so the structure of space is not influenced by anything else; 3. inertial forces can be explained only by accelerations relative to space. We could refuse that space exist in absence of matter (such an idea is counterfactual, therefore non-empirically consistent), or its geometry be Euclidean, or inertial forces have no other explications.
Probably Newton was strongly influenced by the universal opinion for which Euclid's geometry was the only true; only in 19th century other Geometries were proved to be self-consistent. After the discovery of the non-Euclidean geometries the principle according to which space is independent from matter can be considered just one opinion.

The most interesting objections can be summarized in two counter-theories: 1. space is a system of relations between bodies, it has no reality in absence of matter, all movements *including accelerations* are relative to material objects; 2. the conceptual structure of science describes experiences involving observable objects and measurable quantities. Only what we can infer from experience has scientific character, therefore scientists would avoid introducing and making use of any reference to substances, beings, hypotheses etc. without experimental basis.
Both these two points of view had been supported by the Austrian scientist and philosopher *Ernst Mach* (1838 – 1916), whose work had extremely interesting developments in epistemology. According to Mach, Newton's absolute space can not be deduced from experience; the distinction between absolute and relative motion would require the concept of absolute space already from the beginning: *while resting on facts, we know no other than spaces and movements relative to material bodies*. Ptolemaic and Copernican systems are both correct, but the later has proved to be more simple and practical than the first. We have no experience of many universes (e.g. we disagree with the facts while making distinction between Earth in rest with the fixed stars in rotation and Earth in rotation with fixed stars in rest), but of a sole one with its relative movements, the only ones we can measure; according to observation we can affirm only Earth is rotating with respect to the fixed stars, since any other description involves unverifiable hypotheses.

In fact, by analyzing Newton's interpretation we can note a forced conclusion in deriving absolute space by centrifugal forces. Mach himself suggested a possible alternative explication, known as "Mach's principle".

Let's consider Earth's daily motion: we can infer it in different ways, by operating procedures independent each other: for instance, by comparing the period of oscillation of a pendulum at different latitudes, measuring the variation on time of the oscillation plane of the pendulum, the horizontal deflection in free fall, the Coriolis' acceleration…according to Mach, all these observations prove only the motion of Earth relative to fixed stars. We have to inquire into the origin of the centrifugal forces measured by an observer in rotation relative to fixed stars.

Mach suggested solving this problem as follows. While admitting that centrifugal forces manifest accelerations relative to the fixed stars, we implicitly affirm that the origin of inertial forces is the total mass of the far bodies of Universe. Remember that, according to Mach, only interactions between material bodies are significant. Therefore, we are led to consider the forces produced by the remote masses of Universe on bodies in acceleration relative to them. This is a consequent conclusion, since in Dynamics every phenomenon is a manifestation of interactions between physical systems. According to Mach, experiments like Newton's rotating bucket suggest that the source of the inertial forces is the total mass of the Universe. More precisely, acceleration with respect to fixed stars is actually relative to the totality of the bodies further away, which contain the mass of the whole Universe. Therefore inertia is the resultant action of the whole Universe on a single body in acceleration relative to the remote masses, i.e. relative to the average distribution of the matter in the space. This statement is known as *Mach's Principle.* It implies that inertial mass isn't a proper characteristic of each body, but a consequence of its interaction with the whole Universe. In fact it's hypothetical as Newton's space is – we can't perform experiences on a body in absence of the rest of Universe to verify that its inertia vanishes, or to measure directly all the interactions between all the particles of Universe. However, Mach's approach to the problem of inertia suggested the reflections which led Einstein to formulate General Relativity.

6. Inertial and non-inertial observers

To avoid employing unverifiable notions and theories**,** physicists have introduced the concept of *inertial observer*, briefly I.O. The equivalent of a IO in the theory of Newton is an observer in translational uniform motion with respect to space; according to this definition, he's an observer verifying all the laws of Classic Dynamics, in the sense that every phenomenon described by an IO agrees with Newton's laws of motion, the first (Principle of Inertia) in particular. Therefore, the motion of a system not subject to forces is linear uniform with respect to every IO; forces and accelerations measured by IO satisfy the second law $\mathbf{F} = m\mathbf{a}$, etc. On the contrary, non-inertial observers make observations inconsistent with Newton's laws. These disagreements in measures and observations are explained by introducing the concept of *fictional* or *apparent forces* in order to apply the second law to accelerated systems. Inertial forces like the centrifugal are apparent forces since are measured only by non-inertial observers.

To proceed with clarity we have to define the notion of "linear uniform motion" doing without absolute space as reference and without modifying the form of Newton's laws. According to principle of inertia, *by definition* every observer not undergoing external forces is in linear uniform motion, i.e. *by definition* we define linear and uniform the motion of any inertial observer, since inertial observers do not undergo external forces. So we can remove any reference to space or single bodies.

To well understand the problem of inertia, we have to consider the dynamical implications of 2nd law. Velocity and acceleration are introduced in Kinematics, but acceleration is involved in the second Newton's law, $\boldsymbol{F} = m\,\boldsymbol{a}$. Now, according to a reasonable point of view, forces have "objective" nature in the sense that the interactions between physical systems would not depend on the observer (we can agree from a classical point of view, since observations and measures *ideally* restrict themselves to extract information from a physical system without modifying it); for instance, according Newton's law of gravitation, the force of gravity acting on a mass near Earth's surface depends on its position with respect to Earth's centre, so intensity of the force is independent from the motion of the observer, like the daily rotation, etc. In general, from the classical point of view the fundamental forces don't depend on the frame of reference, even if the motion of the observer and the experimental apparatus can influence the measure. Therefore we can admit that *the real value of acceleration is the ratio between force and mass* and consider it as the *absolute acceleration*, without any reference to space etc. Coherently, acceleration is zero in absence of external resultant forces.
People who wish to understand thoroughly the fundamental problems of Dynamics and the birth of Relativity have to focus their attention on measures and observations performed in closed systems and on the role of 2nd law.

How can an observer know if he himself is inertial or not? Let's imagine to be again inside the wagon and suppose that all the bodies not fixed to its walls or to the floor accelerate regardless to the respective masses (in Physics, one says that all the bodies are inside an *acceleration field*), thus – according to the 2nd law – we conclude that each mass m in motion undergoes an apparent force given by $m\,\boldsymbol{a}$; $\boldsymbol{a}$ is the acceleration relative to the observer. By looking outside we observe that speed varies with respect to earth's surface. Thus we conclude that *an acceleration field inside a closed environment in absence of an evident source manifests by itself the acceleration of the environment.* In general, non-inertial observers measure fictial forces and uniform acceleration fields. Such acceleration fields are well distinct from "effective" fields like the gravitational or electromagnetic, since these later are originated by a source and their intensity tends to zero at infinite distance from the source. Instead, inertial forces can manifest the opposite behaviour: centrifugal forces are proportional to the square of the distance from the rotation axis and therefore become infinite at infinite distance.
We can prove that non-inertial observers measure uniform acceleration fields as corollary of first law.

Imagine a perfectly polished sphere put on the perfectly horizontal, smooth floor of an ideal wagon. The sphere is not fixed and no resultant force acts on it. While wagon is in rest or in uniform linear motion, the sphere is in rest with respect to the wagon (the system of reference) and to the observer, who's solidal with the system. If after a certain instant the wagon accelerates, according to the first law the sphere persists in its linear uniform motion and accelerates with respect to the wagon and the observer solidal with it. The acceleration of the sphere *relative to the system of the reference* is obviously opposite in direction and equal in magnitude to the acceleration of the system itself. According to 2nd law, the *inertial force* (apparent) acting on the sphere is $m\,\boldsymbol{a_r}$ [$\boldsymbol{a_r}$ is the acceleration of the sphere relative to the observer]. By denoting with $\boldsymbol{a}$ the "absolute" acceleration, with $\boldsymbol{a_r}$ the one relative to the frame of reference, and with $\boldsymbol{a_s}$ the acceleration of the frame of reference, we get the vector equation

$$\boldsymbol{a_r} = \boldsymbol{a} - \boldsymbol{a_s}$$

If $\boldsymbol{a} = \boldsymbol{0}$, then $\boldsymbol{a_r} = -\boldsymbol{a_s}$: as we have seen before, the relative acceleration of a body not subject to forces is opposite to the drag acceleration of the frame of reference, which in this case appears as an *uniform field*. Even if in the equation of the accelerations $\boldsymbol{a}$ would denote the "real" value, that is the one measured by an inertial observer according to 2nd law, we can formally extend this equation to non-inertial observers measuring accelerations in their own frames of reference. A non-inertial observer considers himself in rest while doing experimental observations, which implies that he observes apparent forces. Therefore he supposes $\boldsymbol{a} = \mathbf{0}$ and again $\boldsymbol{a_r} = - \boldsymbol{a_s}$, which means: all the accelerations $\boldsymbol{a_r}$ relative to a non-inertial observer are opposite to the "real" acceleration $\boldsymbol{a_s}$ of the observer himself – remember that "real" means "measured by an inertial observer", as we got before by correctly employing the formula; the difference is in the interpretation by the non-inertial observer. Since he's virtually in rest, *apparent accelerations are perceived as real*: for example, in a rotating system $\boldsymbol{a_r}$ is perceived as a real *centrifugal force* acting on the rotating observer himself and over the whole reference system, while according to an inertial observer the acceleration is oriented towards the axis of rotation.

Finally we can observe that

All inertial observers are in uniform translational motion with respect to each other
and
Every non-inertial observer is accelerating with respect to all inertial observers.

7. Einstein's Principle of Equivalence

Let's begin defining what a field is. In Classic Dynamics the term "field" denotes properly a quantity depending only on position. For instance, atmospheric temperature is a field ("thermal field") since varies from one point to another. Likewise, *acceleration* of gravity $\boldsymbol{g}$ is a field while *force* of gravity isn't: in fact, $\boldsymbol{g}$ varies depending on position but not on mass since all bodies being in the same position relative to the source mass fall with the same acceleration, while the force of gravity is the product of mass and acceleration and therefore depends also on mass. We correctly define relative accelerations measured in non-inertial systems as "acceleration field" just because are same in the same point, independently on mass. Therefore, the gravitational field is $\boldsymbol{g}$.

Now, consider again experiments performed inside closed environments in acceleration. According to Classical Dynamics, an observer in rest with respect to an accelerated system will detect an acceleration field and infer being non-inertial, but he can also reach a different conclusion. Suppose no way exists to retrieve information from something external. For instance, we can think that the whole Universe is a closed system, since nothing exists outside. Making use only of information from internal environment, as if external world do not exist, and ignoring the difference between inertial and non-inertial systems, an observer will treat the acceleration field as an effective *force field*, real as gravitational or electromagnetic field are. The objection that this point of view results from incomplete information is valid, *if we agree with classical method of treating forces and motion*, but it strongly depends on how we treat information. A logically coherent analysis of this problem leads us to consider acceleration fields as gravitational fields, since they have the same properties of the gravity acceleration $\boldsymbol{g}$, doing abstraction from the source of the field.
Hence no experiment performed inside a system without external information can distinguish an acceleration field due to inertia from a gravitational field.
According to Einstein's *Principle of Equivalence* (1907), we can assume *the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system*. (Einstein 1907).

Many experiments agree with this principle.

1. *plumb-line direction in non-inertial systems*. Imagine being inside a wagon and observe a weight suspended to the ceiling by a rope. While the wagon is in rest with respect to Earth's surface, the direction of the rope is vertical. If the wagon is accelerating uniformly, the direction of the rope is

oblique. The ratio between horizontal projection and vertical is equal to $\frac{a}{g}$ and *not depends on the hanging mass*; *"a"* is the magnitude of wagon acceleration. In fact, the relative acceleration of the weight is the vector sum of $\boldsymbol{a}$ and $\mathbf{g}$: the apparent measure of the gravity acceleration inside the wagon is $\boldsymbol{g} + \boldsymbol{a}$, so $\boldsymbol{a}$ is the horizontal component of the total gravity acceleration. No way is to distinguish between inertial or gravitational effects inside this closed system, *because accelerations relative to the observer are independent from mass*.
[Really, just because of the centrifugal force due to the daily rotation, the plumb line on Earth's surface excluding the poles is not oriented towards the centre, just because the measure of $\boldsymbol{g}$ includes a centrifugal term given by $\omega^2\boldsymbol{r}$ where ω is the angular velocity and $\boldsymbol{r}$ the distance from the axis oriented outwards. At the poles $r = 0$. Therefore the direction of the plumb line is that of the vector $\boldsymbol{g} + \omega^2\boldsymbol{r}$].

2. *Vertical acceleration of a closed system* (e.g. a lift). Consider the following conceptual experiment. An observer is measuring the acceleration of gravity inside a lift. While the lift is in rest or in uniform motion, he will get the usual value of g . If the system speeds up while going up with acceleration a , the observer will measure $g + a$, else if is going down, he'll measure $g - a$. No way is to know if g varies or if the lift is in uniform acceleration on the basis of observations performed inside the lift.

3. *System in free fall*. The observer inside the system doesn't measure any acceleration of gravity, since all surrounding objects fall with the same acceleration and seem in rest. This example is paradoxical, because free fall is the motion of a body undergoing only gravity, but proves *that it is impossible to distinguish between inertial systems in absence of gravity and systems in free fall by exploiting only experimental data obtained in the system itself.*

In a certain sense, the third instance leads us to extend the principle of Classical Relativity to all the systems in free fall, and not only as far as velocity is concerned, but also with regard to acceleration itself. While for non-inertial closed systems in general we can yet recognize the system acceleration by detecting acceleration fields, in the case of free fall it's impossible. We can infer also that *systems in free fall are indistinguishable from the inertial ones*, and therefore consider if gravity itself isn't really a force, since it's impossible detect it inside closed systems in free fall.

The general sense of these considerations is equivalence between inertial and gravitational forces. This results also by the equivalence of inertial and gravitational mass in an implicit form. *Inertial mass* is involved in 2nd law, $\mathbf{F} = m\,\mathbf{a}$. *Gravitational mass* enters the Newton's gravitation law

$$F = G\frac{m_1 m_2}{R^2}$$

The two masses at the numerator are gravitational. However, physicists assume the identity of inertial and gravitational mass, since the ratio between them is a universal constant which we pone equal to 1.

Identity of inertial and gravitational mass implies that **g** *doesn't depend on mass and vice-versa.* In fact, gravity force is given by *inertial mass* times gravitational acceleration, i.e. $W = m_i\, g$ according to the 2nd law, and by

$$W = G\frac{M_g \cdot m_g}{R^2}$$

according to the law of gravitation [M is the source of gravitational field, m the body in free fall. Imagine $M >> m$ and consider both as gravitational mass].

The acceleration g is $\frac{W}{m_i} = G\frac{M_g}{R^2}\cdot\frac{m_g}{m_i}$, which proves it to be independent from mass *if and only if the ratio* $\frac{m_g}{m_i}$ *is same for all bodies*.

Equivalence Principle implies that inertial mass is equal to gravitational mass and vice-versa. Equivalence Principle implies that $\boldsymbol{g}$ is a field, as inertial acceleration $\boldsymbol{a}$ is. If $\frac{m_g}{m_i}$ is not the same for every particle, then $\boldsymbol{g}$ too is depending on the given particle, i.e. isn't a field.

8. **Towards General Relativity**

The Principle of Equivalence itself is still a "classical" principle", but is at once the point of passage to General Relativity. Equivalence of inertia and gravitation implies difference between inertial and non-inertial observer are not fundamental, and we can search for a theory according to which all observers are equivalent. Such a theory must require some relationship between transformations of spatial coordinates and gravitation (really, the transformations involve also time, since Einstein formulated G.R. after Special Relativity). In fact, spinning around an axis z is described by

$$\begin{cases} x' = x\cos\omega t + y\sin\omega t \\ \\ y' = -x\sin\omega t + y\cos\omega t \end{cases}$$

that implies a formal modify of the differential Euclidean metric $\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2$ into a four-dimensional expression as

$$A\,\delta x^2 + B\,\delta x\delta y + C\,\delta y^2 + D\,\delta x\delta t + E\,\delta y\delta t + F\,\delta t^2$$

in which A B etc. denote functions of the coordinates x, y, t . Really, in Special Relativity the invariant interval (generalized distance between two events) is, in four dimensions,

$$\delta s^2 = c^2\,\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$$

or in a more compact form

$$\delta s^2 = \sum_{i=0}^{3}\sum_{j=0}^{3} g_{ij}\delta x^i \delta x^j \text{ or simply } g_{ij}\delta x^i\delta x^j$$

The g_{ij} are the element of the *metric tensor* in which $g_{00} = 1$, $g_{ii} = -1$ if $0 < k \le 3$ and $g_{ij} = 0$ if $i \neq j$. This metric describes in four dimensions the "flat" space-time of the inertial observers. Accelerated motion is described by changes of coordinates transforming the metric. For example, by carrying out the rotation of x and y axes written before and replacing $x\ y\ t$ in the "flat" metric $c^2\,\delta t^2 - \delta x^2 - \delta y^2 - \delta z^2$ with the rotating coordinates $x'\ y'\ t'$ we get (neglect the superscript)

$[c^2\gamma^2 - \omega^2(x^2 + y^2)]\delta t^2 + 2[\omega x\,\delta y - \omega y\,\delta x]\delta t - \delta x^2 - \delta y^2 - \delta z^2$ in which $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, $x^2 + y^2 = R^2$:

it's clear that a spinning reference frame is equivalent to a curvilinear coordinates system. By applying Equivalence Principle, also space-time in a "real" gravitational field must be described in curvilinear coordinates, with the difference that no coordinates transformation can change the curvilinear system associated to a gravitational field into a "flat" system: obviously we can not get a curve space simply by transforming coordinate systems, since curvature is an intrinsic property of n-dimensional varieties. In this way the central idea of G.R. that gravity consists in space curvature came into being.

Final notes

The problem of the mass

The discussion of Einstein's Equivalence Principle has emphasized the intrinsic ambivalence of mass. From a theoretical point of view, it seems separating immediately inertial mass and gravitational mass is more correct, because these two quantities are defined operatively in different ways – their identity is an empirical date. Gravitational mass would be introduced as electric charge is, in the form of "gravitational charge". However, this choice is unusual, according to Newton's method in handling mass as unitary quantity.

Probably, Newton avoided the question by defining mass as the measure of *quantity of matter.* Although vague and not coherent with our knowledge about structure of matter, such a definition has been useful in developing Mechanics. Inertia is a universal property of matter, gravitation too; therefore, we are led "naturally" to unify apparently distinct phenomena exploiting only one quantity. In fact, Newton's work has been successful just by solving the problem of the motion and of the gravitation at the same time.

Inertial systems in Special Relativity

Special Relativity too makes a clear distinction between inertial and non-inertial observers, even stronger than Classical Dynamics. According to Einstein's Principle of Relativity, *the Laws of Physics are invariant in all inertial systems*. It means that equations and formulas expressing fundamental physical laws (like Maxwell's equations) don't change under transformations from an inertial reference frame to another. "*Invariant*" means "same for all inertial observers". This principle is essential in deriving Lorentz's transformations. Accelerating motion is equivalent to a continuum series of inertial observers, each of them with instantaneous velocity $\boldsymbol{v}$ relative to a given inertial frame. As in Classic Dynamics, if a motion is accelerated with respect to a given inertial observer, then it will be even relatively to all inertial observers, but acceleration itself isn't an invariant.

Inertial and non inertial systems in General Relativity

Even G.R. is often defined as a theory according to all reference frames are equivalent, the distinction between inertial and not inertial systems survives in G.R., hidden in the covariant formalism.

GR does not affirm that all reference system are equivalent under every aspect, but only that the laws of Physics have to be formally invariant in all reference frames, i.e. do not change under transformations between reference frames, inertial or non-inertial indifferently.

The strongest form of equivalence between reference systems is Mach's Principle. To clear the problem, consider a system $\boldsymbol{O}$ "in rest" and another $\boldsymbol{P}$ in circular uniform motion with centre in $\boldsymbol{O}$. So even $\boldsymbol{O}$ is in circular uniform motion relative to $\boldsymbol{P}$. Both descriptions have to be perfectly equivalent, since motion is only relative.

In GR, however, there is no perfect equivalence. If $\boldsymbol{O}$ doesn't detect pseudo-gravitational fields (i.e. $\boldsymbol{O}$ is inertial) then at the contrary $\boldsymbol{P}$ does; therefore, $\boldsymbol{P}$ is a system of curvilinear coordinates. Physically it's impossible invert this situation, because a field is something objectively detected. Changes in coordinates are formal (mathematical) transformations, but *choice of coordinates is physically significant* : non-inertial systems are curvilinear coordinates systems. So, $\boldsymbol{O}$ and $\boldsymbol{P}$ are not equivalent, but both can apply GR to experimental data and verify that laws of Physics, in the form stated by GR, are valid.

In GR, the rotation period T' measured in the non-inertial system differs from T measured in the inertial, as follows. Consider the [pseudo]gravitational field in which $\boldsymbol{P}$ is immersed. Time measured in a system inside a [pseudo]gravitational field flows more *slowly*: so, *for an observer in a flat space-time*, physical processes occurring inside the field seem develop more slowly. Therefore, T' is shorter than T. An observer who covers a spatially closed path, starting and ending in the same point in rest relatively to a given inertial frame, will measure a time between start and end shorter than his twin, who has remained in rest relative to the same inertial frame (*Twin Paradox*).

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