ON A. REGHINI'S RECONSTRUCTION OF PYTHAGOREAN GEOMETRY AS AN ALTERNATIVE TO THE HYPOTHESIS OF "GEOMETRIC ALGEBRA"

E.F. Scriptor

ABSTRACT

In 1935, Arturo Reghini - an Italian Mathematics professor and philosopher dedicated to the reconstitution of the esoteric-inspired Pythagorean School - published an essay entitled "*Per la restituzione della geometria pitagorica*", or "For the Restitution of Pythagorean Geometry" in order to explain his point of view on the discoveries and demonstrations traditionally attributed to the original Pythagorean school founded in Crotone (in 'Magna Grecia') in the 6th century BC. His work, like others aimed at the same purpose, was neglected in the academic environment probably also due to the ideological imprint of the author, who was strongly oriented towards esoteric interests and in his last years lived almost in isolation. However, at least part of his treatise should be taken into account as it demonstrates that the achievements of the early Pythagorean school (or those inspired by that in the 5th century BC) could be at all independent of non-Hellenic contributions, such as those attributed to the ancient Babylonians, and should not be interpreted as a 'translation' in the forms of the geometry of solutions of algebraic equations as many have assumed.

The issues addressed in this essay were contained in a book, written in Italian, of which - as far as I know - there are no translations into other languages. There are several more recent editions of the text; for the scholar's *convenience, in the notes I will refer to a faithful version of the paper text freely available on the site E-book campione Liber Liber.*

CONTENTS

THE HYPOTHESIS OF ‘GEOMETRIC ALGEBRA’

In the historiography of Mathematics, the term 'geometric algebra' denotes those parts (problems and proofs) of the ancient geometry of the Greeks, especially in Euclid's Elements, which solve questions that can be solved by algebraic equations and which, in agreement with *O. Neugebauer*, constituted a transposition of algebra into the methods of geometry. According to Neugebauer, followed by more than one scholar, Greek mathematicians rejected the algebraic methods already known to the Babylonians for about a millennium even though they were aware of them, as those were considered insufficient to express exactly irrational quantities such as the square roots of most integers, the golden ratio, etc. “*... there can be no doubt that Greek Mathematics had undergone drastic changes by the time of Plato. The dichotomy between number and continuous magnitude required a new approach to the Babylonian algebra that the Pythagoreans had inherited. The old problems in which, given the sum and the product of the sides of a rectangle, the dimensions were required, had to be dealt with the differently from the numerical algorithms of the Babylonians. A “geometric algebra” had to take the place of the older “arithmetic algebra” and in this new algebra there could be no adding of lines to areas or of areas to volumes…*” [alludes to the lexicon of Babylonian clay tablets, in which the cube, the square of the unknown and the same unknown of the same equation are described as volumes, areas and lengths respectively]. "*From now on there had to be a strict homogeneity of terms in equations, and the Mesopotamian normal forms*, $xy = A, x \pm y = b$ *.*"[1]

The two systems are solved using the equations

$$x\,(b - x) = A \quad \text{(I)}$$

$$x\,(x - b) = A \quad \text{(II)}$$

In both cases the unknown can be the height of a rectangle of given area *A*, whose base is the difference between a given length *b* and the height in the first case, their sum in the second. The Greeks solved these problems by the so-called method of 'application of areas'; there are three cases: the 'simple' application, which solves $bx = A$ (find the height of a rectangle of base *b* and assigned area *A*), the *ellipsis* or 'falling-short' application which corresponds to equation (I) and the *hyperbola* or 'exceeding' application which corresponds to eq. (II). According to *Proclus*[2], also the application of the areas was due to the 'Pythagorean muse' probably referring to Pythagoras himself, or to members of the School of Croton. We do not know whether this attribution was correct, but that is not the point; the question is whether the Greeks solved the problems that lead to (I) and (II) starting from algebra, but

1 C. B. Boyer, *A History of Mathematics 2nd edition* 1991, p. 77.

2 G. R. Morrow, *Proclus – A Commentary on the First Book of Euclid's Elements* 1970.

disguising it by the geometric form, or whether they had already initially used only geometric proofs because they simply were unaware of algebraic methods for solving second-degree equations.

Although it had a wide diffusion among scholars of the last century (perhaps, due to the great authority acquired by Neugebauer thanks to his contributions to the discovery of Babylonian Mathematics), the hypothesis according to which the ancient Pythagoreans or someone else translated the algebra learned directly or indirectly from Mesopotamian civilizations into geometry seems - put precisely in these terms - rather weak, and perhaps scarcely credible[3]. The objections are numerous: no one among the ancient writers mentions a derivation of Greek Mathematics from the Near East (if anything, there has been only talk of Pythagoras' travels to Babylon); between the results obtained by the Babylonians of the ancient kingdom and the Pythagoras' age there is an interval of a millennium, and there are doubts about the preservation of that science through centuries of dramatic political crises, invasions, falls and births of empires[4]; the qualification of 'algebra' given to Babylonian Mathematics is an extension of this term to numerical procedures that solve problems solvable with al-

3 See e.g. J. Høyrup, *What is "geometric algebra", and what has it been in historiography?* 2016.

4 Robson, E., *Influence, ignorance, or indifference? Rethinking the relationship between Babylonian and Greek Mathematics* 2005.

gebra as we understand it today, but we find no formulas that solve the second-degree equation; above all, it is not clear why the Greeks should have ignored those procedures, not difficult and effective, that would have allowed them to easily arrive at the results, if they had known them. It is not clear why they should have learned anything from the Babylonians, from whom they were separated by considerable distances and from other peoples mostly fighting among themselves, and not rather from the Phoenicians and other coastal peoples or even from the Egyptians, with whom they had much more frequent commercial and cultural contacts.

And above all, accepting the hypothesis of geometric algebra as the most probable means assuming that there is nothing better - e.g., a geometric theory, simpler than the one expounded in Euclid's Elements, not so well defined as a deductive system, partly based on deductions, partly on what can be demonstrated by observing a figure (as Socrates does in the *Meno*). Better still, a reconstruction that connects the problem of the application of areas (which greatly interested Neugebauer) to the golden ratio, and to the pentalpha or pentagram, a symbol very dear to the Pythagoreans, and to the construction of the regular pentagon and decagon. All these problems are connected to each other, and if we find a logically correct connection that is likely attributable to the mathematical notions possessed by the Greeks of Plato's time and previous decades (it is not necessary to go back to the School of Croton; the Pythagorean tradition continues in the following centuries) we can accept it as a possible one, perhaps a likely explanation of how Greek mathematicians proceeded before Plato's time.

In fact, a reconstruction that meets these requirements *exists*, and was proposed in the 1930s by A. Reghini, only to be effectively ignored.

Note that I do not intend to assert - nor, I believe, Reghini did - that the Pythagoreans were unaware that the geometric solution to a problem, in which the unknown is a length, can be a method for solving any problem in which the quantity to be calculated is not a geometric quantity. Put otherwise, any problem could be solved geometrically at least in principle. What seems certain to me is that they identified number as the principle of all quantity and relationship between things, and in particular they saw extension as the link between number and quantity. This principle is well illustrated by the theory of figurative numbers. If they had been in possession of a general procedure for solving second-degree equations, the use of geometric methods would have been superfluous, at least for practical purposes. On the contrary, if their research had no utilitarian purposes, as tradition claimed, their attention could be turned to the study of numbers and figures for their properties. This does not rule out the possibility that they were unaware of methods used in geographically nearby civilizations, such as Babylonian algebra (or rather, procedures that allowed the same results to be obtained). Indeed, this ignorance seems highly unlikely; however, it does allow us to affirm that, perhaps precisely because they were dissatisfied with the purely utilitarian nature of the Mathematics then known, they sought its foundations in the close relationship they recognized between number and figure and therefore were interested to deeply study geometry. So they had no incentive to 'translate' algebra into geometry, but to re-establish all of Mathematics, whether they knew the results recorded in the ancient clay tablets discovered in Mesopotamia, handed down to their day, or whether they were unaware of them.

We will only examine the Reghini's solution of the application of areas, the connection with the golden ratio, the construction of the regular pentagon and decagon. It is possible that all this was aimed at constructing the regular dodecahedron, but the safest part as it is mathematically ascertainable is the relationship between the application of the areas (in the case of the hyperbola) and the golden ratio.

In reality, only the case of the hyperbola has a direct relationship with the golden ratio, which may suggest that the starting point of Pythagorean geometry, or rather one of the fundamental themes of its research, was the golden ratio or perhaps, even before that, the construction of the regular pentagon; indeed, it is very likely that this was precisely the direction followed by the Pythagoreans, given that 1. the construction of the regular pentagon is much more difficult than those of the hexagon etc. and *ipso facto* it must have been a goal worthy of attention ; 2. the construction of the golden ratio is a particular case of the exceeding application of areas, and it is likely that, in the initial stages of a science, specific situations are examined before reaching the more general case.

However, R. developed his reconstruction strictly deductively, almost as if the Pythagoreans had applied a method similar to that followed by Euclid. According to this scheme, the applications of the areas precede the construction of the golden section and everything that follows, up to that of the regular dodecahedron. R. assumed that the Pythagorean theorem was already known and that it was sufficient to proceed, but he did not take into account the similarity, even though it must be well known as can be deduced from the interest in proportions and from what tradition attributes to Thales of Miletus. This is probably a mistake, but it does not undermine the logic followed by Reghini.

R. supposed that the Pythagoreans were known the contents of the first 28 Euclid's propositions, that is, what precedes the postulate of parallels and the theory of parallels, and that the Pythagoreans (explicitly or not) admitted: 1st – the postulates of determination and belonging; 2nd – the postulates relating to the division into parts of the line and the plane (reported if wants finite lines and finite planes); 3rd – the postulates of the congruence or movement.

R. considers some basic notions to be demonstrated by the Pythagoreans themselves in an original way and known to them, since ordinary proceedings are obtained from those; precisely they are:

1) the ordinary criteria of equality of triangles;

2) the relationships between the elements of the same triangle; the theorems on isosceles, equilateral, and scalene triangles; the theorem according to which the exterior angle is greater than each of the non-adjacent interiors, the theorem on one side, and

the sum of the other two… [that is, in a triangle each side is less than the sum of the other two].
3) the uniqueness of the perpendicular fom a point to a straight line, the property of perpendiculars to the same straight line, the properties of perpendiculars and obliques, of the axis of a segment, that is, what is essentially obtained with the ordinary postulates and procedures and without the Euclid’s V postulate.[5]

In this essay we will not examine Reghini's entire work, since we exclude the set of notions that he assumed constituted a system of postulates from which to deduce the theorems, proofs such as that of the two-right theorem, the construction of the regular dodecahedron and anything that doesn't concern the golden ratio. The meaning that the Pythagoreans attributed to the pentalpha, to the point of making it the symbol of their sect, the importance of the golden section in the construction of the regular dodecahedron, the originality of Reghini's work on these topics in particular, faced with the lesser (in my opinion) reliability of other parts of his reconstruction, justifies (I think) the choice to deal only with this topic. As regards the Pythagorean theorem, he did not provide an original proof (which is in fact impossible, given the large number of proofs already known), but took into consideration the three simplest

5 *La restituzione…* pp. 26-27.

ones in his opinion[6]: the one proposed by *C. A. Bretschneider* (1870), the one conceived by *Abū al-Ḥasan Ṯābit ibn Qurra,* dead in 901, and the one due to *Bhāskara* (1114-1185), the most recent. Of all these, R. considered the first one due to Pythagoras (according to him, not to the school in general, but to the founder himself).

6 *La restituzione…* pp. 56-57.

APPLICATIONS OF AREAS: REGHINI'S EXPLANATION

Proposition I.44 of the *Elements* poses the following problem. "*To a given straight line to apply, in a given angle, a parallelogram equal to a given triangle*", so introducing the method of application of areas. Commenting on this very proposition, *Proclus* attributed the authorship of the method to the Pythagoreans. "*There would seem to be no reason to doubt that the particular solution, like the whole theory, was Pythagorean, and not a new solution due to Euclid himself*", *Heath* stated, in his commentary on the Elements.[7]

Propositions II.5 and II.6 can be interpreted respectively as the geometric solutions of the equations[8]

$$ax - x^2 = b^2 \quad \text{and} \quad ax + x^2 = b^2$$

The topic is taken up again, in much greater depth, in VI. 27-28-29; in VI. 30 Euclid sets forth the construction of the golden section.

R. assumed that in this part of geometrythe Pythagoreans always proceeded without regard to the theory of parallels, similarity, proportions, and the two postulates of Euclid and Archimedes. As with the similarity, doing without proportions

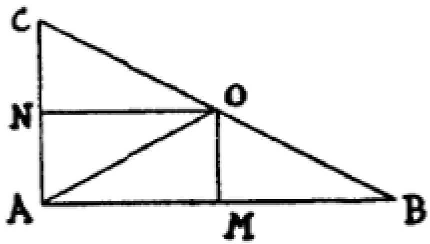
7 *T. L. Heath*, *The Thirteen Books of Euclid's Elements*, 2nd ed., vol. I 1956 p. 343.

8 *Heath* p. 383ff. However, these developments seem a forcing of the letter of Euclid's text.

seems a gratuitous and unjustified choice, especially considering that the Pythagoreans were very interested in them. Therefore, Reghini's reconstruction is longer and more artificial than it would be if he had made use of similarity.

To solve, after that of the simple application (‘parabola’), the other two problems of the application, R. begins with the following theorem and its inverse:[9]

THEOREM: The midpoint of the hypotenuse of a right angled triangle is equidistant from the three vertices, and conversely, if in a triangle the midpoint of one side is equidistant from the three vertices, it is right-angled.



Starting from A , in the half-plane having the line AB as its margin and containing C, we draw the half-line which forms an angle equal to $A\hat{B}C$ with AB, which intersects the hypotenuse BC in O. The triangle AOB is therefore isosceles with respect to the base AB , and consequently AO = OB. We

9 The proofs will be reported with some variation for reasons of clarity. The drawings reproduce the original ones of the online version.

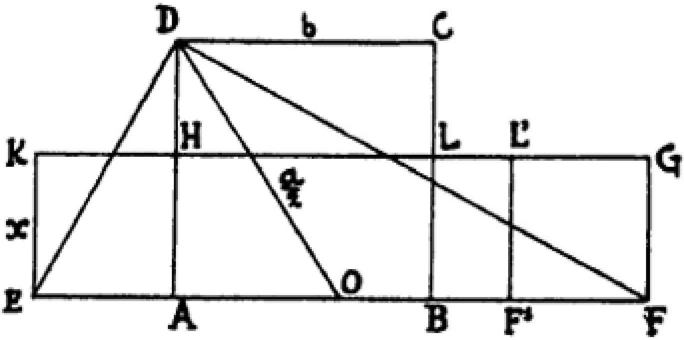
must now prove that OB = OC. The angles $C\hat{A}O$ and $A\hat{C}O$ are equal because they are complementary to equal angles $A\hat{B}C$ and $O\hat{A}B$, so the triangle $C\hat{A}O$ is isosceles on the base AC etc.

Evidently, R. assumed that the Pythagoreans had not made use of the theorem attributed to Thales by the ancient tradition, according to which the semicircle inscribes a right angle, and therefore the midpoint of the diameter is equidistant from the endpoints and the vertex of the right angle. Perhaps he took it for granted that the Italic school was completely independent from the Ionian one, even though Thales preceded Pythagoras by a few decades, but it is possible that he was influenced by ideological reasons that led him to exclude *a priori* any external contribution to the Italic school.

Conversely, if in a triangle ABC is O the midpoint of BC and OA = OB = OC, we get $O\hat{A}C = O\hat{C}A$ and $O\hat{A}B = O\hat{B}A$, since by the two-right theorem the sum of these four angles is equal to two right angles, we will have: $O\hat{A}C + O\hat{A}B$ = a right angle.

Now let's move on to the 'falling-short' application. Given a square of side b and a length a, we have to construct a rectangle equivalent to the square such that the sum of its sides be equal to a.[10]

10 R. used the same drawing for the *short-falling* and the *exceeding* application.

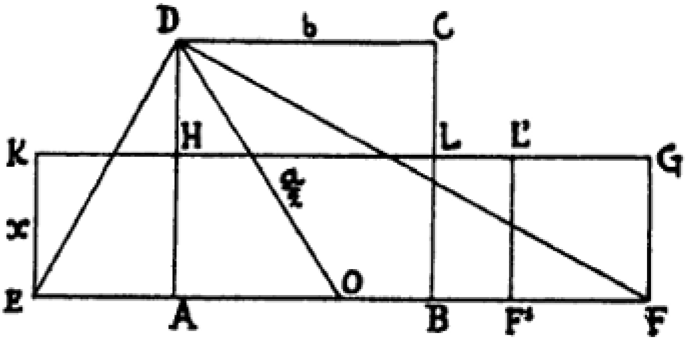


Let ABCD be the square of side AB = b. Taken on the straight line AB on the side of A the point O such that DO is equal to half of a, let's choose on AB the points E and F such that OE = OD = OF ; by the previous theorem the triangle EDF is right-angled; and therefore the square constructed on the height AD is equal to the rectangle of sides AF, AE. Constructed the rectangle EKGF , with EK = AE , removing from it the rectangle AHGF equivalent to the square ABCD , the difference AEKH is precisely a square. The rectangle AHGF then solves the problem, and EA = AH = EK is the x of the equation

$$x\,(a-x) = b^2$$

For the problem to admit a real solution it must be $a > 2b$.

Let us now proceed with the 'exceeding' application. In this case, we need to construct a rectangle equivalent to a square with a given side b , knowing that the difference between its sides is a given length a. Following R. we will reason on the same drawing:



As in the previous case, we construct the square ABCD with side b. Extend AB from the B side up to point F' so that AF' = a. Then join the midpoint O of AF' with D, and on the line AB take points E and F on opposite sides with respect to O so that OE = OD = OF. The triangle EDF is right-angled at D, so for the Euclid's second theorem the square ABCD is equivalent to the rectangle with sides AH = EA and AF. The segments F'F and EA are equal because they are differences of equal segments, OF = OE and OF' = OA, therefore EF' = AF. The solution rectangle is thus EF'L'K. If the unknown x is the width of the rectangle, we have solved

$$x\,(a+x) = b^2\,.$$

A weak point, perhaps too weak, of the procedures here presented is the use of Euclid's second theorem. This is not immediately derivable from the Pythagorean one, and in fact R. demonstrates it starting from that through a series of passages that take up an entire page,[11] moreover explained with the use of literal expressions, that is, not directly by synthetic means. Instead, it is very easy to prove Euclid's first and second theorems from the similarity relations between a right-angled triangle and the two right-angled triangles obtained by dividing it by the height relative to the hypotenuse; in VI. 8, Euclid applied this strategy, and in VI. 31 he proved that "*In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle*", that is a generalized version of the Pythagorean theorem. *Tannery* was also of the same opinion,[12] and in this regard he pronounces himself as follows: "Proclus clearly states that the proof [of the Pythagorean theorem] due to Euclid is precisely his; but Pythagoras' is absolutely unknown to us, and it is a useless effort to seek the most primitive reasoning to attribute it to him. *Pythagorean Geometry was already advanced enough for it to be demonstrated by means of similar triangles*."

Whatever the proofs known to the early Pythagoreans of the theorems known as the Pythagorean and Euclid theorems, the connection between the case of the hyperbola and the golden ratio is clear; the construction of the latter is a special case of the former. Speaking in the abstract, it may be - following R. - that the second one was reached starting from the first, but the opposite could have happened, considering that the relevance of the golden ratio in geometry is far superior to the applica-

11 *Per la restituzione…* p. 64.

12 P. Tannery, *La géométrie grecque* 1887.

tions of areas. Furthermore, the construction of the regular decagon can be done starting from the isosceles triangle having angles of 36°, 72° and 72° and this can - as we will see thanks to the work of R. - be traced without knowing its relationship with the golden ratio. Or, the golden section was already known and only later its connection with the regular decagon and pentagon was discovered; it is not possible to reconstruct exactly the historical succession of the oldest advances in geometry.

THE GOLDEN SECTION

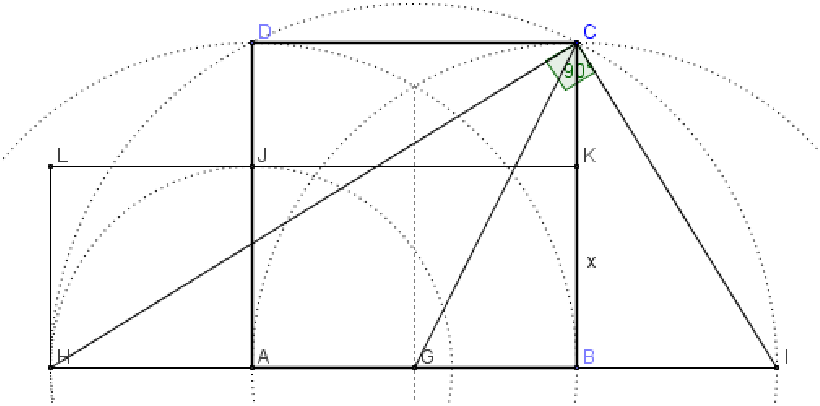
In geometry, the problem is to divide a segment so that the square, the side of which is the largest part (golden part), be equal to the rectangle having as sides the entire segment and the remaining part. According to Reghini's reconstruction, it can be traced back to the problem of 'exceeding' application in the case where the difference between the sides of the searched rectangle (a) is equal to the side b of the square.

Reghini's method is an improvement, in terms of simplicity and clarity, on that adopted by Euclid in VI. 30, itself an immediate consequence of proposition 29, a variant of application by excess: "*To a given straight line to apply a parallelogram equal to a given rectilinear figure and exceeding by a parallelogrammic figure similar to a given one.*" A notable case – the one we are interested in – asks to extend a segment of length a by an unknown length x so that the rectangle of base $a + x$ and height x is equivalent to a square of side a. If $b = a$, we fall in the case

$$x^2 + ax = a^2 \quad \text{or} \quad x^2 = a\,(a - x)$$

treated by Euclid in VI. 30, and x is the golden section of a.

As with the applications of areas, R. starts from the right-angled triangle to geometrically solve the equation $x^2 = a\,(a - x)$.



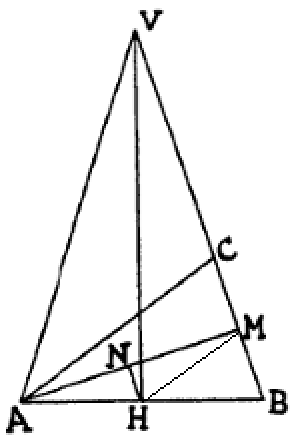
Given the square ABCD of side a , we must apply Euclid's second theorem to a right-angled triangle that we construct by joining the vertex C with the points H and I from opposite parts of the midpoint G of AB and such that GC = CH = CI . Let’s construct the rectangle HBKL with height BK = HL equal to HA, which in turn is equal to BI as differences of equal segment; its sides are equal to the projections of the legs onto the hypotenuse and therefore HBKL is equivalent to the square ABCD . They have the rectangle ABKJ in common, so the uncommon parts, the square HAJL and the rectangle JKCD , are equivalent. Let BK $= x$, we get $x^2 = a\,(a-x)$.

THE CONSTRUCTION OF THE REGULAR PENTAGON

The most straightforward way to construct the regular pentagon is through the regular decagon, whose side is the golden ratio of the radius (distance from the vertices to the center of the circle circumscribed around the decagon). There are several ways to directly draw the pentagon; moreover both constructions can be done starting from the isosceles triangle ('notable triangle') with a vertex opposite the base equal to 36°. In fact, the regular decagon is the union of ten 'notable' triangles, and 'notable' triangles are those formed by the diagonals of a regular pentagon emerging from the same vertex and the side opposite to it.

The 'remarkable' triangle is such because the side opposite the angle of 36° is the golden ratio of those opposite the angles of 72°. This property can be demonstrated through similarity; but we have seen that R. believes that the Pythagoreans did not use it, presumably because the theory of similarity, if deductively developed, presupposes that of parallels according to Euclid, that is, starting from the fifth postulate, which R. (correctly) considers alien to the notions possessed by the Pythagoreans; therefore R. systematically tried to formulate proofs that dispensed with similarity. In my opinion, it is not possible to argue, solely on the basis of considerations based on respect for logical coherence, that similarity was unknown to the first Greek mathematicians, a position frontally contradicted by what tradition attributed to Thales, and completely unreasonable considering the practical usefulness of similarity relations (scale projects, indirect measures, etc.). Reghini's research is not important on a historical level, but rather because it demonstrates that everything leading to the pentagon, the pentalpha and then the regular dodecahedron can be obtained only on the basis of rules of equivalence of areas and Pythagorean theorem.

R. managed to show that "*The base of an isosceles triangle having the vertex angle equal to the fifth part of two right angles is the golden part of the side*" without similarity, which is the key point of his entire procedure.

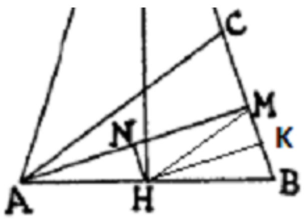


The bisector AC divides VAB into two triangles VAC , isosceles on the base VA, and ACB isosceles on BC, so VC =

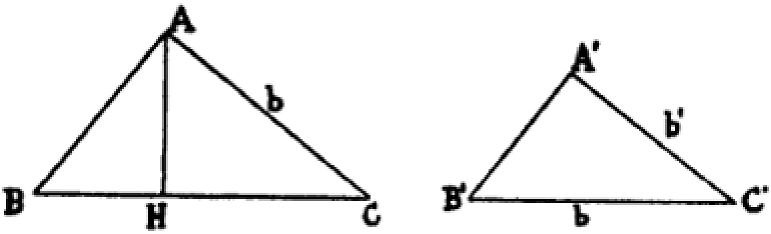
AC = AB . Furthermore VAB and ACB have neatly equal angles. Let's draw the heights VH and AM of the triangles VAB and ABC respectively, and the height HN of the isosceles triangle AHM ; we get

$$NH = \frac{1}{2} BM = \frac{1}{4} BC$$

For the first of the two equalities, draw the height of MHB which is isosceles on BM , and note that HKMN is a rectangle:



Right-angled triangles VAH, AHN have angles equal, and the leg AH of the first is the hypotenuse of the second one. At this point, R. invokes a corollary of Euclid's first theorem for which "*If two right-angled triangles have equal angles and a leg of one of them is equal to the hypotenuse of the other, the square built on the leg of the first is equal to the rectangle whose sides are the hypotenuse of the first and the homologous leg of the second.*"



In fact, $AC^2 = BC \cdot HC$ and $HC = A'C'$.

The problem with this whole reconstruction is that R. must deduce both of Euclid's theorems from Pythagoras', which is possible but quite laborious.[13]

Based on the above corollary, we can conclude that the square of AH is equivalent to the rectangle of sides VA e NH :

$$AH^2 = VA \cdot NH$$

13 In a note on the method he adopted, R. expressed himself in this way: "LORIA (*Scienze esatte* p. 41) attributes to Pythagoras the construction of the isosceles triangle with the angle at the vertex half that of the base, bringing it back to the construction of the golden part; but to demonstrate that the base is the golden part of the side he resorts to the similarity of the triangles VAB, ABC... and he seems to mean that this way was also kept by Pythagoras. The development we have shown starts, instead, from the Pythagorean theorem, and uses only consequences of this theorem... and the problems of the application that we know they had been solved by the Pythagoreans."

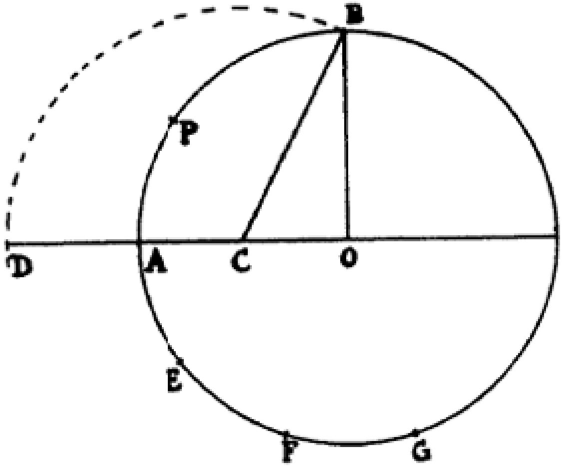
I will report faithfully in the appendix the proofs of Euclid's theorems attributed to the Pythagorean school by R., in an English translation of the original Italian version.

where AH is a leg of VHA and the hypotenuse of ANH ; NH is the homologous of AH in ANH .

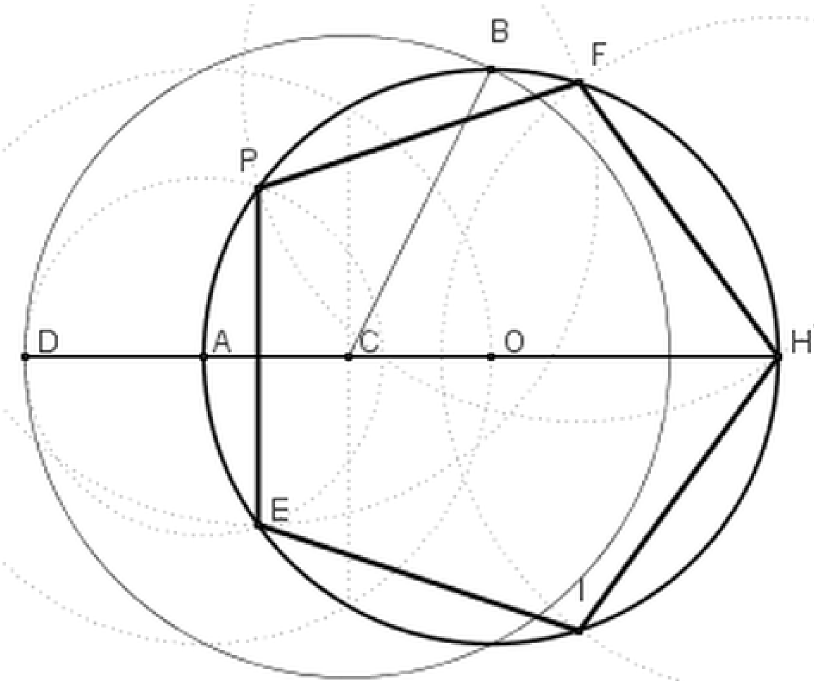
It follows that the square of AB is equivalent to the rectangle of sides VB = VA and BC = 4 NH, that is, AB is the golden section of the oblique side of VAB .

R. then proves the converse theorem.

At this point, the rest is relatively trivial. Among the various ways to proceed, R. proposed a construction of the regular decagon inscribed in a circle; below I report the original drawing.



I propose a more complete construction, based on the number value of the golden ratio, that is $\frac{\sqrt{5}-1}{2}$. Assume that the radii of the crf Γ of center O are unitary, that $AC = CO = \frac{1}{2}$, so $CB = \frac{\sqrt{5}}{2}$; construct the crf of center C and radius CB that intersects in D the extension of AO on the side of A . The segment $DA = \frac{\sqrt{5}-1}{2}$ is the golden section of the radius. Plot the crf of center A and radius DA , which intersects Γ in P and E , so $AP = AE$ is the side of the decagon. Join P with E and get the side of the pentagon. Apply the segment PE to Γ until you get back to E and have drawn the regular pentagon: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The main merit of the research conducted by R. was to identify a series of logical derivations from a set of notions present to Euclid, excluding his theory of parallels with all consequences including similarity. We have already seen that this working hypothesis is a claim that is difficult to sustain even for common sense reasons, given that it is at least risky to assume that the Pythagoreans did not use similarity only because they had not developed a theory of parallels. Such a line of inquiry would presuppose that the Pythagoreans' investigations were conducted according to extremely rigorous deductive methods, which requires an advanced stage of Mathematics, and in any case is not a situation that occurs universally; indeed, it was fully realized in Greece following decades-long efforts, while R. himself seems to accept the idea that Pythagoras and his followers started from a very narrow set of notions. However, if we accept that at least Euclid's theorems had been proved by similarity, many unnecessarily difficult and historically unlikely steps would be significantly made simpler, and so modified the general scheme appears much more plausible.

It is possible, however, that over time the epigones of the school of Croton - meaning by this term those who defined themselves as 'Pythagoreans' after the mid-fifth century, that is, after the dispersion of the sect - the need for a more rigorous arrangement had made them reflect on the desirability of replacing intuitive and logically incomplete proofs, which employed similarity as originating from the equality of form and the proportions connected to it, with proofs founded on the Pythagorean theorem and its consequences, on the theorem of the two right angles and in general on Euclid's first 28 propositions, as R. himself admitted; in fact Euclid also separates what can be proved without similarity from what requires it, explicitly enunciating the fifth postulate and deducing the rules of similarity from a theory of parallels based on that postulate. The idea, maintained by R., of having to do without all the consequences of the fifth postulate would therefore be correct on the level of logical rigor, but it seems more difficult to support it on the historical one, given that we should not exclude *a priori* the recourse to intuition and the use of notions that appear evident and easily employable.

Another weakness is the assumption that the golden ratio was treated as a notable case of the 'exceeding' application. In reality, the proof offered by R. himself does not require the precedent ones at all, and the original meaning of the golden ratio itself lies not in the theory of equivalences, but in proportions. R. must have followed Euclid's *Elements* and Proclus' account in his attempt to logically order a matter in the process of formation and therefore consider the golden ratio as a consequence of applications of areas.

Where Reghini's work fully achieves this goal is the demonstration of the possibility that Greek geometry developed autonomously from the very beginning through the study of figures and their properties regardless of practical applications. It is not logically necessary to hypothesize that the applications of areas are a disguise for algebraic equations; above all - and truly fundamental - is the success of producing a construction of the golden ratio and the regular pentagon from the Pythagorean theorem and a few other elementary notions, likely within the reach of mathematicians prior to Plato's time.

APPENDIX

A literal translation of Reghini's derivation of Euclid's two theorems from the Pythagorean one.

THEOREM: The square constructed on the height of a right-angled triangle is equal to the rectangle having as its sides the projections of the legs on the hypotenuse. Let AH (fig. 16) be the height of the right-angled triangle ABC.

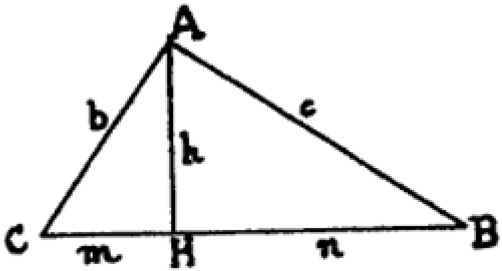

Fig. 16

And let m, n be the CH, HB projections of the two legs. For convenience, indicating rectangles and squares with modern notations (but without introducing the concepts of proportion and measure with this), from the right-angled triangle ABC we have:

$m^2 + h^2 = b^2$ and therefore:

$m^2 + h^2 + c^2 = b^2 + c^2$

On the other hand,

$a = m + n$

therefore:

$m^2 + n^2 + 2mn = a^2$

but

$b^2 + c^2 = a^2$

therefore also:

$m^2 + h^2 + c^2 = m^2 + n^2 + 2mn$

and for the second common notion:

[α] $h^2 + c^2 = n^2 + 2mn$

but

$c^2 = h^2 + n^2$

and therefore:

$h^2 + c^2 = 2h^2 + n^2$; $2h^2 + n^2 = n^2 + 2mn$; $2h^2 = 2mn$

[β] $h^2 = mn$

Having proved this theorem, we observe that the second member of [α] is the sum of two rectangles having the same height n and the bases n and 2m; it is therefore equal to the rectangle of base n + 2m, and height n, that is:

$n^2 + 2mn = n(n+2m) = h^2+c^2$

or also:

$n(n+m) + nm = h^2 + c^2$

and for [β]

$n(n+m) = c^2$

that is

$na = c^2$

We therefore have the theorem:

THEOREM: The square constructed on a leg of a right-angled triangle is equal to the rectangle whose sides are the hypotenuse and the projection of the leg above the hypotenuse. This is Euclid's so-called first theorem. Let us remember that Proclus attests us that the theorem is not due to Euclid and that Euclid only owns the exhibition found in the text of the Elements (Book I, 47). In Euclid, the proof is based on the parallel postulate; from this we then obtain the Pythagorean theorem, and from the two the other theorem thus called by Euclid. The following corollary immediately follows from this theorem. COROLLARY: If two right-angled triangles are equiangular to each other and one side of one of them is equal to the hypotenuse of the other, the square constructed on the side of the first is equal to the rectangle whose sides are the hypotenuse of the first and the homologous side of the second.

FINAL NOTES ON A. REGHINI.

A. Reghini (Florence, 1878 – Budrio, near Bologna, 1946) is best known as an eminent exponent of Italian esotericism. His works are still published by publishing houses oriented towards that sector; in reality, his original education was scientific and specifically mathematical (he graduated in Mathematics in 1912 from the University of Pisa) and for some time taught Mathematics in secondary schools. He attempted to revive the ancient Pythagorean School, sharing the traditional idea that the study of arithmetic and geometry was the path that led to a higher understanding of reality. He had no relevance in the academic field, and probably for this reason, and for his interests and political orientation, and perhaps for his very limited interest in fame, his contribution to the study of Pythagorean Mathematics was almost completely ignored even by specialists in that field.

I felt I should publish a partial exposition of his treatise both because the neglect of this historian of Pythagoreanism is objectively a gap that should be filled; at least, it certainly does no harm to historical research to consider the conclusions of those who have studied a subject for years, much more than eminent historians have done and who in his own right should be considered one of the greatest specialists in ancient Greek Mathematics. I have expressed my doubts about important points of his reconstruction, but I do not see that more eminent scholars than R. have provided much better: we must accept that almost only hypotheses can be formulated on all the Mathematics of the period preceding Plato, and among all these Reghini's reconstruction has if nothing else the merit of clarity and logical coherence.

The author is a private scholar with a degree in Physics.

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